DIRAC's NOTATION (bracket notation)

The definite integral over all space of an operator sandwiched between two functions occurs often, and various abbreviations are used:

$$\int f_m^* \hat{A} f_n \, d\tau \equiv \langle f_m | \hat{A} | f_n \rangle$$ Dirac's notation

For the definite integral over all space between two functions, we write

$$\int f_m^* f_n \, d\tau \equiv \langle f_m | f_n \rangle$$

It is trivial to prove the identity $\langle f_m | f_n \rangle = \langle f_n | f_m \rangle^*$

HERMITIAN OPERATORS

The QM operators must be not only linear but also Hermitian (postulate 2). Let $\hat{A}$ be a linear QM operator. The average value of the property A is

$$\langle A \rangle = \int \psi^* \hat{A} \psi \, \mathrm{d}\tau$$

Where ψ is the state function of the system. Since the average value of a physical quantity must be a real number, we demand that

$$\langle A \rangle = \langle A \rangle^* \qquad \int \psi^* \hat{A} \psi \, \mathrm{d}\tau = \int \psi \left(\hat{A} \psi\right)^* \mathrm{d}\tau$$

A linear operator that satisfies this equation for all well-behaved functions is called **Hermitian operator**. Note that the above equation must hold not only for ψ wave functions

HERMITIAN OPERATORS (an equivalent definition)

If an operator is linear (we know that QM operators are linear) then its "Hermitianity" can be also expressed as

$$\int f^* \hat{A} g \, d\tau = \int g (\hat{A} f)^* \, d\tau$$

for all well-behaved functions *f* and *g*. In Dirac's notation:

$$\langle f | \hat{A} | g \rangle = \langle g | \hat{A} | f \rangle^*$$

The two definitions are equivalent (if the operator is linear … indeed QM operators are linear)

$$\int f^* \hat{A} g \, d\tau = \int g (\hat{A} f)^* \, d\tau \leftrightarrow \int \Psi^* \hat{A} \Psi \, d\tau = \int \Psi (\hat{A} \Psi)^* \, d\tau$$

The eigenvalues of a Hermitian operator are real numbers

HERMITIAN OPERATORS

Two functions f and g of the same set of coordinates are said to be orthogonal if

$$\int f^* g \, d\tau \equiv \langle f | g \rangle = 0$$

where the integral is a definite integral over the full range of the coordinates

Theorem

Two eigenfunctions of a Hermitian operator $\hat{A}$ that correspond to different eigenvalues are orthogonal; eigenfunctions of $\hat{A}$ that belong to a degenerate eigenvalue can always be chosen to be orthogonal (Schmidt orthogonalization)

(Lev.5 p. 167-169)

NORMALIZATION of EIGENFUNCTIONS

An eigenfunction can usually be multiplied by a constant to normalize it, and we shall assume, unless stated otherwise, that all eigenfunctions are normalized:

$$\int f^* f \, d\tau \equiv \langle f | f \rangle = 1$$

NORMALIZATION of EIGENFUNCTIONS

An eigenfunction f of a QM operator can be normalized as follows

Be $\langle f|f\rangle \neq 1$

We define $g = c\,f$ where c is a real number. If f is eigenfunction of $\hat{A}$, then also g is eigenfunction of $\hat{A}$ (already shown)

We want that $\langle g|g\rangle = 1$

This implies that $\int c^* f^* c\, f\, d\tau = 1$ ⇨ $c = \dfrac{\pm 1}{\sqrt{\int f^* f\, d\tau}}$

The choice of the sign of c is arbitrary

Therefore the eigenfunctions of an Hermitian operator are **orthonormal**

$$\langle f_i | f_j \rangle = \delta_{ij}$$

$$\delta_{ij} = 1 \quad \text{if} \;\; i = j$$

Kronecker delta

$$\delta_{ij} = 0 \quad \text{if} \;\; i \neq j$$

EXPANSION of a FUNCTION in TERMS of EIGENFUNCTIONS

A set of functions $g_1, g_2, \dots, g_i, \dots$ is said to be a **complete set** if any well-behaved function f that obeys the same boundary conditions as the g_i's can be expanded as a linear combination of the g_i's

$$f = \sum_i a_i \, g_i$$

where the a_i's are (complex) constants. Of course, it is understood that f and the g_i's are all functions of the same set of variables

EXPANSION of a FUNCTION in TERMS of EIGENFUNCTIONS

POSTULATE

If $\hat{A}$ is any linear Hermitian operator that represents a physically observable property, then the eigenfunctions of $\hat{A}$ form a complete set

COMMUTATION THEOREMS

THEOREM

If two linear operators have a common complete set of eigenfunctions, then the operators commute

(Lev.5 p. 176)

$$[\hat{A}, \hat{B}] = \hat{0}$$

THEOREM

If two Hermitian operators commute, we can select a common complete set of eigenfunctions for them