$C_t - D\Delta C = 0 \quad \text{in } \Omega_e(t)$

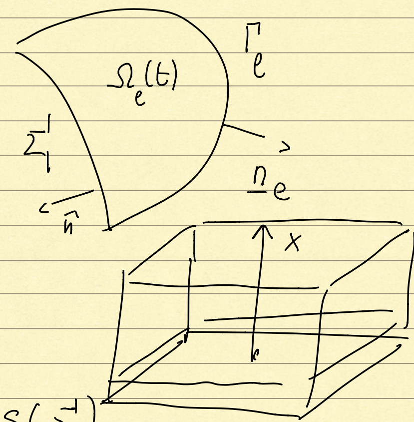
$C(\underline{x},0) = \hat{C}(\underline{x})$

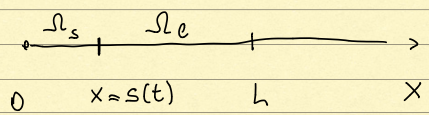
$\nabla C \cdot \underline{u}_e = 0$

$S(\underline{x},0) = \hat{S}(\underline{x}) = 0$

$\rho \dfrac{S_t}{\|\nabla S\|} = k(C - C_0)_+ \quad \text{su } S^1\ (\Sigma_1)$

$-D\nabla C \cdot \hat{n} + C \dfrac{S_t}{\|\nabla S\|} = -k(C - C_0)_+ \quad \text{su } S(\Sigma_1)$





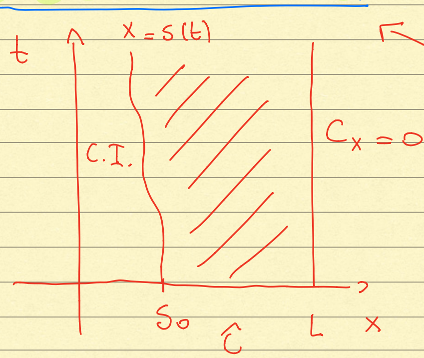
$S(x,t) = 0$

$S = x - s(t) = 0$

$\nabla S = 1 \qquad S_t = -\dot{s}$

Pma 1D

$$\begin{cases} C_t - DC_{xx} = 0 & t \geq 0 \quad x \in [s(t), L] \\ C(x,0) = \hat{C}(x) & x \in [s_0, L] \\ s(0) = s_0 & s_0 \in (0, L) \\ C_x(L,t) = 0 & t \geq 0 \\ -\rho \dot{s} = k(C - C_0)_+ & \text{su } x = s(t) \quad t \geq 0 \\ -DC_x - C\dot{s} = -k(C - C_0)_+ & \text{su } x = s(t) \quad t \geq 0 \end{cases}$$



Posso sostituire l'ultima condizione con

$$-DC_x(s,t) = c(s,t)\dot{s} + \rho\dot{s}$$

$$-DC_x(s,t) = \dot{s}\,[c(s,t)+\rho]$$

FORMA ADIMENSIONALE del pb.ma 1D

$x = \tilde{x}L \qquad s = L\tilde{s} \qquad t = t_{ref}\,\tilde{t}$ $\qquad t_{ref}$ da specificare

$$c = c_A\tilde{c} \qquad c_A = \max_{[s_0,L]} \hat{c}(x)$$

$$c_t - Dc_{xx} = 0 \qquad \left(\frac{\cancel{c_A}}{t_{ref}}\right)\tilde{c}_{\tilde{t}} - \left(\frac{D\cancel{c_A}}{L^2}\right)\tilde{c}_{\tilde{x}\tilde{x}} = 0$$

$$\tilde{c}_{\tilde{t}} - \frac{t_{ref}}{(L^2/D)}\,\tilde{c}_{\tilde{x}\tilde{x}} = 0 \qquad t_D = \frac{L^2}{D}$$ tempo caratt. di diffusione

$$\boxed{\tilde{c}_{\tilde{t}} - \left(\frac{t_{ref}}{t_D}\right)\tilde{c}_{\tilde{x}\tilde{x}} = 0}$$

$$\boxed{\tilde{c}(\tilde{x},0) = \frac{\hat{c}(\tilde{x}L)}{c_A} = \tilde{\hat{c}}(\tilde{x})}$$

$$\boxed{\tilde{c}_{\tilde{x}}(1,\tilde{t}) = 0}$$

$$\boxed{\tilde{s}(0) = \tilde{s}_0 = \frac{s_0}{L}}$$

Adesso voglio vedere come si riscalano le condizioni su $\tilde{x} = \tilde{s}(\tilde{t})$

$$-\rho \dot{s} = K(c - c_0)_+$$

$C_0 / C_A = \delta \qquad \delta \ll 1$

$$\left(-\frac{\rho L}{t_{ref}}\right) \dot{\tilde{s}} = K C_A \left(\tilde{c} - \cancel{C_0 / C_A}^{\,10^{-5}}\right)_+$$

Se supponiamo $C_A = 10^{-2}$ mol/et

$C_0 = 10^{-7}$ mol/et

$\delta = 10^{-5} \ll 1$

$$\lambda = \left(\frac{\rho}{C_A}\right)$$

$$-\frac{\lambda}{t_{ref}} \dot{\tilde{s}} = \frac{1}{\underbrace{(L/K)}_{\rightsquigarrow\, t_R}} \tilde{c}(\tilde{s}, \tilde{t})$$

$t_R = \left(\frac{L}{K}\right)$ tempo caratt. della reazione

$$\boxed{-\frac{\lambda}{t_{ref}} \dot{\tilde{s}} = \frac{\theta}{t_D} \tilde{c}(\tilde{s}, \hat{t})}$$

1° cond. su $\tilde{s}$

$$\theta = \frac{t_D}{t_R} \qquad \frac{1}{t_R} = \frac{\theta}{t_D}$$

$$\dot{s}\,[c(s,t) + \rho] = -D\, C_x(s,t)$$

$$\frac{L \cancel{C_A}}{t_{ref}} \dot{\tilde{s}} \left[\tilde{c}(\tilde{s}, \hat{t}) + \lambda\right] = -D \frac{\cancel{C_A}}{L} \tilde{c}_{\tilde{x}}(\tilde{s}, \hat{t})$$

$$\boxed{\frac{\dot{\tilde{s}}}{t_{ref}} \left[\tilde{c}(\tilde{s}, \tilde{t}) + \lambda\right] = -\frac{1}{t_D} \tilde{c}_{\tilde{x}}(\tilde{s}, \tilde{t})}$$

2° condizione

Il pb.mo adimensionale è (ometto i ~)

$$\boxed{\begin{cases} C_t - \left(\dfrac{t_{ref}}{t_D}\right) C_{xx} = 0 \\ C(x,0) = \hat{C}(x) \\ C_x(1,t) = 0 \\ -\dfrac{\lambda}{t_{ref}} \dot{s} = \dfrac{\theta}{t_D} c(s,t) \\ \dfrac{\dot{s}\,[c|_s + \lambda]}{t_{ref}} = -\dfrac{1}{t_D} C_x(s,t) \\ S(0) = S_0 \end{cases}}$$

$$\theta = \frac{t_D}{t_R} = \frac{L^2/D}{L/K} = \frac{LK}{D}$$

Da questo problema ne ottengo 2 diversi a seconda di prendere

$t_{ref} = t_D$ (Scala diffusiva)

$t_{ref} = t_R$ (" reattiva)

(PD) $t_{ref} = t_D$

$$\begin{cases} C_t - C_{xx} = 0 & [s(t),1] \quad t \geq 0 \\ C(x,0) = \hat{C}(x) \\ C_x(1,t) = 0 \\ -\lambda \dot{s} = C|_s \cdot \theta \\ \dot{S}\,[C|_s + \lambda] = -C_x|_s \\ S(0) = S_0 \end{cases}$$

(PR)

$$\begin{cases} \theta C_t - C_{xx} = 0 & [s(t),1] \quad t \geq 0 \\ C(x,0) = \hat{C}(x) \\ C_x(1,t) = 0 \\ -\lambda \dot{s} = C|_s \\ \theta \dot{S}\,[C|_s + \lambda] = -C_x|_s \\ S(0) = S_0 \end{cases}$$

Pma scala diffusiva

Pma scala reattiva

Tecniche di p.to fisso per provare $\exists$ e $!$ di PD e PR (consideriamo solo PD)

Considero la trasf.

$$\boxed{v(x,t) = \lambda(x - s(t)) + \int_{s(t)}^{x} c(q,t)\,dq}$$

$$U_t = -\lambda \dot s - c(s,t)\dot s + \int_s^x c_t(q,t)\,dq \qquad c_t = c_{xx}$$

$$U_t = \underbrace{-\lambda \dot s - c(s,t)\dot s}_{=0} + c_x(x,t) \underbrace{- c_x(s,t)}_{=0} \qquad \boxed{U_t = c_x}$$

$$U_x = \lambda + c(x,t)$$

$$\boxed{U_{xx} = c_x} \longrightarrow \boxed{U_t = U_{xx}}$$

U soddisfa l'eq.ne del calore

$$c_x(1,t) = 0 \implies U_t(1,t) = 0 \qquad U(1,t) = \text{cost.}$$

$$\implies U(1,t) = U(1,0)$$

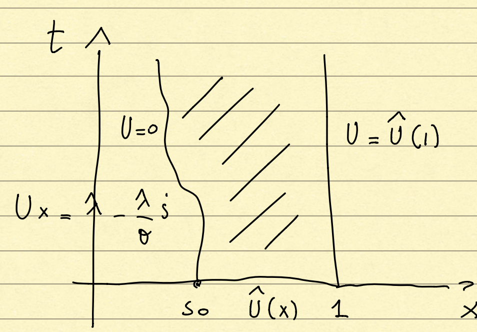
$$\boxed{U(x,0)} = \lambda(x - s_0) + \int_{s_0}^x \hat c(q)\,dq = \boxed{\hat U(x)} \implies \boxed{U(1,t) = \hat U(1)}$$

Le condizioni di frontiera per U

$$\boxed{U(s,t) = 0}$$

$$\boxed{U_x(s,t) = \lambda + c(s,t) = \left(\lambda - \frac{\lambda}{\theta}\dot s\right)}$$

$$(\mathrm{Pb}\,U)\quad \boxed{\begin{cases} \boxed{\begin{array}{l} U_t = U_{xx} \\ U(x,0) = \hat U(x) \\ U(1,t) = \hat U(1) \\ U(s,t) = 0 \end{array}} \; (*) \\ U_x(s,t) = \lambda - \dfrac{\lambda}{\theta}\dot s \\ \boxed{s(0) = s_0} \end{cases}}$$



Considero un insieme Z costituito da funzioni $s(t)$ con una certa regolarità t.c. $s(0) = s_0$! Prendo s^* in Z e risolvo il pb.ma

$$\begin{cases} u_t = u_{xx} \\ u|_0 = \hat{u}(x) \\ u(1,t) = 0 \\ u(s^*, t) = 0 \end{cases}$$

PDE parabolica $\rightsquigarrow$ $u(x,t;s^*)$

Uso l'altra condizione di frontiera

$$\begin{cases} \dfrac{\lambda \dot{s}}{\theta} = \left[\lambda - u_x(s^*, t; s^*)\right] \\ s(0) = s_0 \end{cases}$$

Da questo problema ottengo una nuova $s(t)$. Allora ho costruito una mappa

$$\begin{aligned} \mathcal{T} : Z &\longrightarrow Z \\ s^* &\longmapsto \tilde{s} \end{aligned}$$

Se $\mathcal{T}$ ha un p.to fisso $\bar{s}$ $\Rightarrow$ questo è soluzione del mio pbmo

$$u(x,t;\bar{s}) \qquad \bar{s}$$

Se tale p.to è unico $\Rightarrow$ la soluzione è unica !!

Cerchiamo la soluzione stazionaria di (PD) e (PR)

Pb.mo stazionario

$$\begin{cases} C_{xx} = 0 \\ C_x\big|_1 = 0 \\ C\big|_{s_\infty} = 0 \\ C_x\big|_{s_\infty} = 0 \end{cases}$$

s_∞ soluzione stazionaria

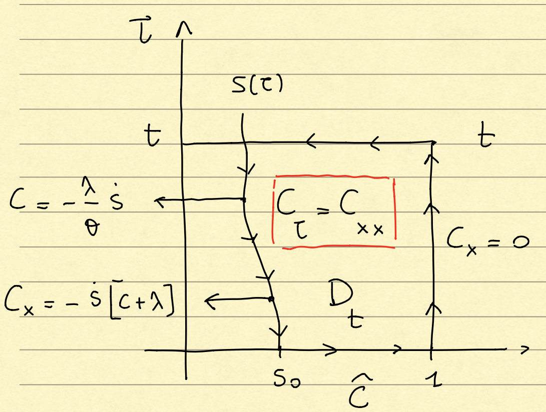
$$s_\infty = \lim_{t \to +\infty} s(t)$$

la solu. è $C_\infty \equiv 0$

$$s_\infty = ??$$

Questo pb.mo non mi dice chi è s_∞ (??)

FORMULAZIONE INTEGRALE per la frontiera mobile $s(t)$



$$D_t = \left\{ (x,\tau) : \quad \tau \in [0,t] \quad x \in [s(\tau), 1] \right\}$$

$$\iint_{D_t} \left[C_\tau - C_{xx} \right] d\tau \, dx = 0$$

$$\iint_{D_t} \left[\frac{\partial}{\partial \tau}(c) - \frac{\partial}{\partial x}(C_x) \right] dx \, d\tau \underset{\substack{\text{formule} \\ \text{di Gauss} \\ \text{Green}}}{=} - \oint_{\partial D_t} c \, dx + C_x \, d\tau = 0$$

$$\begin{cases} dx = \dot{s} \, d\tau \\ d\tau = d\tau \end{cases}$$

$$\Rightarrow \quad 0 = \int_{s_0}^{1} \hat{c}(x) \, dx + \underbrace{\int_0^t \cancel{C_x(1,\tau)} \, d\tau}_{=0} - \int_{s(t)}^{1} c(x,t) \, dx - \int_0^t c\big|_s \, \dot{s} \, d\tau - \underbrace{\int_0^t C_x\big|_s \, d\tau}$$

$$C_x(s,t) = -\dot{s}\left[c\big|_s + \lambda\right]$$

$$0 = \int_{s_0}^{1} \hat{c}(x)\,dx - \int_{s}^{1} c(x,t)\,dx - \int_0^t \cancel{c(s,\tau)}\dot{s}\,d\tau + \int_0^t \dot{s}\left[\cancel{c(s,\tau)} + \lambda\right]d\tau$$

$$\lambda\left[s(t) - s_0\right] = \int_{s(t)}^{1} c(x,t)\,dx - \int_{s_0}^{1} \hat{c}(x)\,dx$$

Se faccio $\lim_{t \to \infty} s(t) = s_\infty$ $\lim_{t \to \infty} c(x,t) = c_\infty = 0$

$$\lambda\left[s_\infty - s_0\right] = -\int_{s_0}^{1} \hat{c}(x)\,dx$$

$$\boxed{\lambda s_\infty = \underbrace{\int_0^{s_0} \lambda\,dx}_{\text{mosso molare iniziale di solido}} - \underbrace{\int_{s_0}^{1} \hat{c}(x)\,dx}_{\text{mosso molare iniziale di } H^+}}$$

$s_\infty > 0$ (Ho più solido che H^+ e quindi non consumo tutto il solido)

$s_\infty < 0$ (Non sufficiente solido per far reagire tutti gli H^+. Questo vuol dire che in un tempo finito si esaurisce tutto il solido)