$$\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1 \qquad \lim_{x \to 0} \frac{\ln(1+\sin x)}{\sin x} = 1$$

$$\lim_{x \to x_0} \frac{\ln(1+f(x))}{f(x)} = 1 \implies$$ se x è abbastanza vicino a x_0

$$\ln(1+f(x)) \sim f(x)$$

$f(x) \to 0$ per $x \to x_0$

$$\lim_{x \to 0} \frac{\ln(1+\sin x)}{\sin x} \cong \lim_{x \to 0} \frac{\sin x}{\sin x} = 1$$

LIMITI NOTEVOLI CON IFINITESIMI, INFINITI

$f(x)$ si dice infinitesimo per $x \to x_0$ se $\lim_{x \to x_0} f(x) = 0$

$$\lim_{x \to x_0} \left(1 + t f(x)\right)^{\frac{1}{f(x)}} = e^t \qquad \lim_{x \to x_0} \frac{\ln(1+f(x))}{f(x)} = 1$$

$$\lim_{x \to x_0} \frac{e^{f(x)} - 1}{f(x)} = 1 \qquad \lim_{x \to x_0} \frac{\sin(f(x))}{f(x)} = 1 \qquad \lim_{x \to x_0} \frac{(1+f(x))^{\alpha} - 1}{f(x)} = \alpha$$

$g(x)$ è infinito per $x \to x_0$ se $g(x) \to +\infty$ per $x \to x_0$

$\frac{1}{g(x)} \to 0$ $\qquad$ $\frac{1}{g(x)}$ è infinitesimo $\qquad$ $\left(1 + \frac{1}{x}\right)^x$

Di nuovo allora $\lim_{x \to x_0} \left(1 + \frac{t}{g(x)}\right)^{g(x)} = e^t$

$$\lim_{x \to x_0} \frac{\ln\left(\frac{1}{g(x)} + 1\right)}{\frac{1}{g(x)}} = 1 \quad \ldots\ldots$$

Se $f(x)$ è infinitesimo $\Rightarrow$

$$\boxed{\begin{aligned} &\ln(1+f(x)) \sim f(x) \\ &e^{f(x)} \sim 1 + f(x) \\ &\sin(f(x)) \sim f(x) \\ &(1+f(x))^{\alpha} \sim 1 + \alpha f(x) \end{aligned}}$$

Ricordandosi che

$$\lim_{x \to 0} \frac{(1-\cos x)}{x^2} = \frac{1}{2} \quad \Rightarrow \quad \lim_{x \to x_0} \frac{(1-\cos(f(x))}{f(x)^2} = \frac{1}{2}$$

$$1 - \cos(f(x)) \sim \frac{f(x)^2}{2} \iff \boxed{\cos(f(x)) \cong 1 - \frac{f(x)^2}{2}}$$

ESERCIZI

$$\lim_{x \to -1} \frac{(x+1)\sin(\pi x)}{1 - \cos(x+1)} = \begin{bmatrix} t = (x+1) \\ x \to -1 \\ t \to 0 \end{bmatrix} = \lim_{t \to 0} \frac{t \sin(\pi(t-1))}{1 - \cos t} = *$$

$$\sin(\pi t - \pi) = -\sin(\pi t) \qquad (*) = \lim_{t \to 0} \frac{-t\sin(\pi t)}{1 - \cos t} = \lim_{t \to 0} \frac{-\cancel{t} \cdot \pi \cancel{t}}{\cancel{t^2}/2} = -2\pi$$

Es.

$$\lim_{x \to -\pi/2} \frac{\sin x + 1}{(2x+\pi)^2} = \begin{bmatrix} \text{sost.} \\ 2x + \pi = z \\ z \to 0 \end{bmatrix} = \lim_{z \to 0} \frac{1 + \sin\left(\frac{z-\pi}{2}\right)}{z^2} = (*)$$

$$\sin\left(\frac{z}{2}-\frac{\pi}{2}\right) = \sin(z)\cancel{\cos(\pi/2)} - \sin\left(\frac{\pi}{2}\right)\cos\left(\frac{z}{2}\right) = -\cos\left(\frac{z}{2}\right)$$

$$(*) = \lim_{z\to 0} \frac{1-\cos(z/2)}{z^2} =$$

$$= \lim_{z\to 0} \frac{(\cancel{z}/2)^2 \cdot \frac{1}{2}}{\cancel{z^2}} = \frac{1}{8}$$

$$\left(1-\cos(f(x)) \simeq \frac{f(x)^2}{2}\right.$$

Es

$$\lim_{x\to 2} \frac{\sin(3^x-9)}{\log(x^2-3)} = \lim_{x\to 2} \frac{3^x-9}{x^2-4} = \left(\begin{array}{l}\text{Sost.}\\ x = t+2\\ 3^x = 3^t\cdot 9\end{array}\right) = (*)$$

$\boxed{f(x) = 3^x - 9}$ è infinit. per $x \to 2$

$$3^t = \left(e^{\ln 3}\right)^t = e^{t\cdot\ln 3}$$

$$\boxed{\sin(f(x)) \sim f(x)}$$

$$g(x) = 1+(x^2-4) \implies \boxed{\log(1+x^2-4) \sim x^2-4}$$

$$\ln 3$$

$$(*) = \lim_{t\to 0} \frac{3^t 9 - 9}{(t+2)^2-4} = \lim_{t\to 0} \frac{9(3^t-1)}{t^2+4t} = \lim_{t\to 0} \frac{9(3^t-1)}{t} \cdot \frac{1}{(t+4)} =$$

$$(\quad) = \frac{e^{\ln 3\cdot t}-1}{t} \simeq \frac{\ln 3\cdot\cancel{t}}{\cancel{t}} = \ln 3$$

$$= \left(\frac{9\ln 3}{4}\right)$$

$$\lim_{x\to 0} \frac{a^x-1}{x} = \ln a$$

Es

$$\lim_{x\to 0^+} x^{\sin x} =$$

$$\boxed{\begin{array}{l}\text{Siccome } \sin x \text{ è infinitesimo per } x\to 0\\ \sin x \sim x\end{array}}$$

$$= \lim_{x\to 0^+} x^x = \lim_{x\to 0^+} \left(e^{\ln x}\right)^x = \lim_{x\to 0^+} e^{x\cdot\ln x} = e^{0^-} = 1^- = 1$$

$\times$

Devo calcolare $\lim_{x \to 0^+} x \cdot \ln x = \left(\begin{array}{l} \text{sost. } x = \frac{1}{t} \\ x \to 0^+ \\ t \to +\infty \end{array} \right) = \lim_{t \to +\infty} \frac{1}{t} \cdot \ln\left(\frac{1}{t}\right) =$

$= \lim_{t \to +\infty} \frac{\cancel{\ln(1)} - \ln(t)}{t} = \lim_{t \to +\infty} - \frac{\ln(t)}{t} = 0^-$

Es

$$\lim_{x \to -\infty} \frac{e^{x^2}}{\left(1 + \frac{1}{x}\right)^{x^2}} = \frac{+\infty}{1^{\infty}} = \lim_{x \to -\infty} \frac{e^{x^2}}{e^{\ln(1+\frac{1}{x}) \cdot x^2}} = (*)$$

$\ln(1 + f(x)) \sim f(x)$ se $f(x)$ è infinitesimo

$\ln\left(1 + \frac{1}{x}\right) \sim \frac{1}{x}$ se $x \to \pm\infty$

$$(*) = \lim_{x \to -\infty} \frac{e^{x^2}}{e^{\frac{1}{x} \cdot x^2}} = \lim_{x \to -\infty} \frac{e^{x^2}}{e^{x}} = \lim_{x \to -\infty} e^{x^2 - x} =$$

$= e^{+\infty} = +\infty \rightsquigarrow$ risultato del limite.

$$\lim_{x \to +\infty} \frac{\sqrt[5]{x^5 + 2x^3} - x}{\sin(1/x)} = \frac{x \sqrt[5]{1 + \frac{2x^3}{x^5}} - x}{\sin(1/x)} = \lim_{x \to +\infty} \frac{x\left(\sqrt[5]{1 + \frac{2}{x^2}} - 1\right)}{\left(\frac{1}{x}\right)}$$

$\sin(f(x)) \sim f(x)$

$\sin\left(\frac{1}{x}\right) \sim \frac{1}{x}$ per $x \to +\infty$

$$= \lim_{x \to +\infty} \left[\frac{\overbrace{\left(1 + 2/x^2\right)^{\frac{1}{5}} - 1}}{\frac{2}{x^2}} \right] \cdot 2 = (++)$$

$g(x) = \frac{2}{x^2}$ ē infinitesimo per $x \to +\infty$

$$(1+g(x))^{\alpha} - 1 \sim \alpha g(x) \qquad \boxed{\left(1+\frac{2}{x^2}\right)^{\frac{1}{5}} - 1 \sim \frac{1}{5} \cdot \frac{2}{x^2}}$$

$$(++) = \lim_{x \to +\infty} \frac{\cancel{\left(\frac{2}{x^2}\right)} \cdot \frac{1}{5}}{\cancel{\left(\frac{2}{x^2}\right)}} \cdot 2 = \left(\frac{2}{5}\right) \qquad \frac{20}{5} = 4 \qquad \frac{2}{5} = \frac{4}{10} = 0.4$$

Es.

$$\lim_{x \to 0} \left[\frac{1}{\frac{x \sin x}{\cos x}} - \frac{1}{x \sin x} \right] = \pm(\infty - \infty) =$$

$$= \lim_{x \to 0} \frac{1}{x \sin x} \left[\frac{1}{\left(\frac{1}{\cos x}\right)} - 1 \right] = \lim_{x \to 0} \frac{1}{x \sin x} \left[\cos x - 1 \right] =$$

$$= \lim_{x \to 0} - \frac{(1 - \cos x)}{x \sin x} = - \lim_{x \to 0} \frac{\frac{x^2}{2}}{x^2} = -\frac{1}{2}$$

Es.

$$\lim_{x \to \frac{\pi}{2}} \left(1 + \cos^2 x\right)^{\tan^2 x} = 1^{\infty}$$

$\cos^2 x$ ē infinitesimo per $x \to \frac{\pi}{2}$ $\qquad \boxed{(1+f(x))^{\frac{1}{f(x)}} \to e}$ se $f(x)$ ē infinitesimo

$$\lim_{x \to \frac{\pi}{2}} \left(1+\cos^2 x\right)^{\frac{\sin^2 x}{\cos^2 x}} = \lim_{x \to \frac{\pi}{2}} \boxed{\Bigg[\underbrace{\left(1+\cos^2 x\right)^{\frac{1}{\cos^2 x}}}_{\to e}\Bigg]^{\sin^2 x}} = e$$

Es.

$$\lim_{x \to 0} \frac{e^{\tan^3 x} - 1}{x\left(\cos x - e^{x^2}\right)} = \lim_{x \to 0} \frac{\cancel{1} + x^3 - \cancel{1}}{x\left(\cancel{1} - \frac{x^2}{2} - \cancel{1} - x^2\right)} = (*)$$

Ricordiamo che se $f(x)$ è infinitesimo

$$e^{f(x)} - 1 \simeq f(x) \qquad \Longleftrightarrow \qquad \boxed{e^{f(x)} \simeq 1 + f(x)}$$

$$1 - \cos f(x) \approx \frac{f(x)^2}{2} \qquad \qquad \boxed{\cos(f(x)) \simeq 1 - \frac{f(x)^2}{2} \quad (*)}$$

$\tan^3 x$ è infinitesimo
x è infinitesimo
x^2 " "
per $x \to 0$

Se uso $(*)$

$$\begin{cases} e^{\tan^3 x} \simeq 1 + \tan^3 x \simeq 1 + x^3 \\ e^{x^2} \simeq 1 + x^2 \\ \cos x \simeq 1 - x^2/2 \end{cases}$$

$\frac{\tan x}{x} \to 1$ se $x \to 0$ $\qquad$ $\tan x \simeq x$ se $x \to 0$

$$(*) = \lim_{x \to 0} \frac{\cancel{x^3}}{\cancel{x}\left(-\frac{3}{2}\cancel{x^2}\right)} = -\frac{2}{3}$$