EXERCISES FOR THE COURSE
FUNCTIONS OF COMPLEX AND HYPERCOMPLEX VARIABLE

Exercise 1. Prove that an entire function having a pole at \( \infty \) is a polynomial.

Exercise 2. Show that each of the functions \( e^z, \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \) and \( \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \) has an essential singularity at \( \infty \).

Exercise 3. Consider any rational function
\[
f(z) = \frac{P(z)}{Q(z)},
\]
where \( P(z), Q(z) \) are coprime complex polynomials; let \( \deg(f) := \max\{\deg(P), \deg(Q)\} \). What is the general form of \( f \) supposing \( f \) maps \( \partial \Delta(0,1) \) into itself and:
1. \( \deg(f) \leq 1 \)
2. \( \deg(f) \) is arbitrary
3. \( \deg(f) = 1 \) and \( f \) maps \( \Delta(0,1) \) into itself?

For each case listed, is the corresponding class of functions a group with respect to composition? Why?

Exercise 4. Compute the complex derivative of
\[
M_w(z) := \frac{z - w}{1 - \overline{w}z}
\]
and check that \( M_w'(w) = \frac{1}{1 - |w|^2} \).

Exercise 5. For all \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}) \), we have defined the transformation \( F_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) by the formula
\[
F_A(z) := \frac{az + b}{cz + d}.
\]
Prove that \( F_A \) preserves \( \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \) if, and only if, \( A \in GL(2, \mathbb{R}) \); and that, in such a case, \( F_A \) maps the upper half-plane into itself if, and only if, \( \det A > 0 \).

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